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Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

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ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF INITIAL-BOUNDARY VALUE PROBLEMS FOR A DISPERSIVE HYPERBOLIC EQUATION

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Prepared for
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Abstract

Initial-boundary value problems for an energy conserving dispersive hyperbolic equation, the Klein-Gordon equation, are considered. This equation exhibits the main feature of dispersion: The speed of propagation depends on frequency. Problems in two space dimensions with a parabolic boundary are discussed.

The primary purpose of this paper is to compare the asymptotic expansion of solutions obtained by a technique we call the *ray method* with the asymptotic expansion of the exact solution. In the cases considered, the solutions agree.

In addition a numerical comparison is made of the exact and asymptotic solutions for a specified region of space time.

Asymptotic Expansions of Solutions of Initial-Boundary Value Problems for a Dispersive Hyperbolic Equation

N. Bleistein and Robert M. Lewis

(1) Introduction

In the past few years there has been a great deal of interest in the asymptotic expansion of solutions of boundary value problems and initial-boundary value problems for partial differential equations. In some cases, the "far field" is obtained by asymptotic methods and in other examples the expansion is obtained for large values of a parameter that appears "naturally" in the equation, and/or initial and boundary conditions.

A large class of problems can be treated by methods in which certain curves, called rays, play an essential role. Their importance rests on the fact that on these rays the functions which are used to make up the solution can be shown to satisfy ordinary differential equations which are often solvable.

This "ray method" has been extensively studied by J. B. Keller and his associates at the Courant Institute.* A significant advantage of the method is that it eliminates the necessity of first finding the exact solution in order to find the asymptotic expansion. Consequently, many problems for which the exact solution is not known can be treated by the ray method. In particular it has led to a "geometric theory of diffraction" [2].

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* D. S. Jones [1] refers to this method as "Keller's Method."

Recently, the ray method has been applied to time dependent problems and, in particular, dispersive hyperbolic equations [3]. A particular equation of this type, the Klein-Gordon equation, may be used to describe the propagation of electromagnetic waves in certain plasmas [3]. Initial-boundary value problems for this equation are presently being studied, since it is the simplest example of a large class of dispersive hyperbolic equations.

As of now, there is no general proof that the ray method actually yields the asymptotic expansion of the exact solution. The purpose of the research in this paper is to compare the leading term of the asymptotic expansion of the exact solution and the "ray solution" for particular problems. The results are shown to agree and thereby add to the overwhelming evidence that the ray method is valid.

We consider initial-boundary value problems in two space dimensions for the Klein-Gordon equation. The boundary is a parabola and we require that the solution satisfy a homogeneous condition.

The total solution is assumed to consist of a sum of a primary field and a reflected field. The associated rays are called primary and reflected rays and are straight lines in space time. The primary field must satisfy the equation and the initial conditions. The reflected field is taken to be zero initially and is determined at the boundary in such a way that the total solution satisfies the given boundary condition. The primary rays are emitted from the plane $t = 0$ in space-time. For each primary ray incident on the boundary, a reflected ray is produced. We introduce the notion of group speed and a group velocity vector for the rays. These quantities are "frequency dependent" due to the dispersive nature of the equation being solved.

Two types of problems are considered. The distinguishing feature is the way in which the initial data depends on the large parameter. In one case, at most two primary rays can emanate from each point in the "initial plane," t = 0. In the other, an infinite set of primary rays are produced at some points of the initial plane. Their group speeds cover the complete continuum of values from zero up to the "characteristic speed" of the equation (see figure 1). The "ray method expansion" of the leading term of the primary and reflected fields are then calculated.

These problems have been so chosen that the solution can also be obtained exactly by time-reducing the equation and solving a partial differential equation in the space variables. This new equation is solvable by using a method introduced by H. Lamb [4] to treat the reduced wave equation in a domain with a parabolic boundary. The solution, u, is expressed as a Fourier transform of the solution of the time-reduced problem. This transform can then be expanded asymptotically by standard methods of asymptotic expansion of integrals. In both cases, the solutions obtained by the two methods are seen to agree exactly.

For a particular example, we make a numerical comparison of the far fields of the exact and asymptotic solutions. The results are almost identical for large values of the parameter and agree surprisingly well for smaller values.

(2) Solution by the Ray Method

(2.1) Summary of the Ray Method

Asymptotic solutions for $\lambda \to \infty$ of the hyperbolic partial differential equation*

$$(1) \qquad c^2 u_{x_\nu x_\nu} - u_{tt} - \lambda^2 b^2 u = 0$$

are discussed in [3]. The results obtained there are similar to those discussed in [6], but are considerably simpler because b and c are constants in (1). In [3] asymptotic solutions of (1) of the form,

$$(2) \qquad u(t,\underline{X}) \sim \exp[i\lambda s(t,\underline{X})] z(t,\underline{X}); \quad \underline{X} = (x_1, \cdots, x_n),$$

are obtained. In this section we summarize the results derived there.

By inserting (2) in (1) one obtains a <u>transport equation</u> for z and a <u>dispersion equation</u> for s. These partial differential equations may then be solved explicitly. In order to describe the solutions s and z, we introduce the quantities

$$(3) \qquad k_\nu = s_{x_\nu} \quad \omega = -s_t \quad \underline{K} = (k_1 \cdots, k_n); \quad k^2 = k_\nu k_\nu;$$

which satisfy the <u>dispersion relation</u>.

$$(4) \qquad c^2 k^2 - \omega^2 + b^2 = 0, \text{ or } \omega = h(k) = \pm h_o(k); \; h_o = \sqrt{c^2 k^2 + b^2} \; .$$

* Repeated indices are summed from 1 to n.

We also introduce an n-parameter family of straight lines in $(t,\underline{X})$-space. These lines are called rays*. In terms of the n parameters $\underline{\Sigma} = (\sigma_1,\cdots,\sigma_n)$ the rays are given by $\underline{X} = \underline{X}(t;\underline{\Sigma})$ where

$$(5) \qquad x_\nu(t;\underline{\Sigma}) = x_{\nu o}(\underline{\Sigma}) + [t-t_o(\underline{\Sigma})]\, g_\nu(\underline{\Sigma}).$$

Here g_ν is a component of the group velocity vector $\underline{G} = (g_1,\cdots, g_n)$ defined below. Along each ray $\underline{K}$ and ω have the constant values

$$(6) \qquad k_\nu = k_{\nu o}(\underline{\Sigma}),\ \omega = \omega_o(\underline{\Sigma}),$$

which must satisfy (4). Furthermore

$$(7) \qquad g_\nu = g_\nu(\Sigma) = \frac{\partial\omega}{\partial k_\nu} = \frac{c^2 k_\nu}{\omega} = \frac{c^2 k_{\nu o}(\underline{\Sigma})}{\omega_o(\underline{\Sigma})}\ .$$

Along each ray, $s(t,\underline{X})$ is given parametrically by

$$(8) \qquad s = s(t) = s[t,\underline{X}(t;\ \underline{\Sigma})] = s_o(\underline{\Sigma}) + [t-\tau(\underline{\Sigma})]\,\ell(\underline{\Sigma}),$$

where

$$(9) \qquad \ell = k_\nu g_\nu - \omega = \frac{c^2k^2}{\omega} - \omega = -\frac{b^2}{\omega} = -\frac{b^2}{\omega_o(\underline{\Sigma})}\ .$$

Furthermore, $z(t,\underline{X})$ is given parametrically by

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* The lines in space-time and their projections in space are called rays. The meaning will be clear in context.

$$(10)\quad z = z(t) = z[t,\underline{X}(t;\underline{\Sigma})] = z_o(\underline{\Sigma})\left[\frac{j(t_o)}{j(t)}\right]^{1/2}\;;\quad z_o(\underline{\Sigma})=z[t_o,\underline{X}_o(\underline{\Sigma})].$$

Here $j(t) = j(t;\underline{\Sigma})$ is the jacobian of the mapping defined by the rays (5), that is

$$(11)\qquad j(t;\underline{\Sigma}) = \det \frac{\partial x_\nu(t;\underline{\Sigma})}{\partial \sigma_\mu}\ .$$

We see that the function (2,8,10) is uniquely determined once the "initial values" t_o, $\underline{X}_o$, s_o, $\underline{K}_o$, ω_o, and z_o are given as functions of $\underline{\Sigma}$. In general, the values of t_o, $\underline{X}_o$, s_o and z_o on some "initial manifold" M in $(t,\underline{X})$space are derived from the data (initial data, boundary data, etc.) of the given problem for (1). The values of $\underline{K}_o$ and ω_o then follow from (3) and (4). For example, if M is the initial plane $t = 0$, and $z(0,\underline{X}) = z_o(\underline{X})$ and $s(0,\underline{X}) = s_o(\underline{X})$ are known, we may set $x_{\nu o} = \sigma_\nu$. Then

$$(12)\qquad x_{\nu o} = \sigma_\nu,\ t_o = 0,\quad s_o = s_o(\underline{\Sigma}),\ z_o = z_o(\underline{\Sigma}).$$

By differentiation along M we find that*

$$(13)\qquad k_{\nu o} = \frac{\partial s_o}{\partial x_\nu}(\underline{\Sigma}),\quad \omega_o = \pm\sqrt{c^2\frac{\partial s_o}{\partial x_\nu}\frac{\partial s_o}{\partial x_\nu} + b^2}\ .$$

The values of $s_o(\underline{\Sigma})$ and $z_o(\underline{\Sigma})$ may be determined from the initial data of the given problem,(see section (2.2).). If M is a boundary, given

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* Since(4) is quadratic, there are two solutions for ω_o. We shall see that, in general, the asymptotic solution will be a sum of two terms of the form (2), one for each value of ω_o.

parametrically by

$$(14)\qquad x_\nu = x_{\nu o}(\sigma_1, \cdots, \sigma_{n-1}),\ t = \tau\ (\tau = \sigma_n)$$

the values

$$(15)\qquad s(\tau, \underline{X}_o) = s_o(\underline{\Sigma});\quad z(\tau, \underline{X}_o) = z_o(\underline{\Sigma})$$

are obtained from the boundary data of the given problem. (See section (2.4).) Then by differentiation along M we find that

$$(16)\qquad \omega_o = -\frac{\partial s_o}{\partial \tau},\quad k_{\nu o}\frac{\partial x_\nu}{\partial \sigma_j} = \frac{\partial s_o}{\partial \sigma_j};\quad j = 1, \cdots, n-1.$$

Then (16) and (4) provide n+1 equations for the n+1 quantities $\omega_o, k_{1O}, \cdots, k_{nO}$.

In some cases the initial manifold M is of dimension less than n. In section (2.3), n = 2 and M is the (one dimensional) line

(17) $$x_1 = x_o (= \text{constant}),\ t = 0,\ x_2 = \sigma_2;$$

while, on M,

(18) $$s(0, x_o, \sigma_2) = s_o (= \text{constant}).$$

Differentiation along M yields

(19) $$k_2 = k_{20} \equiv 0.$$

Then (19) and (4) provide only two equations for the three quantities k_{10}, k_{20}, ω_0. We may therefore introduce an additional independent parameter, say $k_{10} = \sigma_1$, and then $\underline{K}_o$ and ω_o are given by

(20) $$k_{10} = \sigma_1,\ k_{20} = 0,\quad \omega_o = \pm\sqrt{c^2\sigma_1^2 + b^2}\ .$$

If, as in the above example, the initial manifold is of dimension less than n, it is easy to see that M is a "caustic" of the ray family, i.e. the jacobian j(t) vanishes on M. In this case the solution (10) for z(t) must be modified because $z(t_o)$ is infinite. In [3] it is shown that

(21) $\qquad z = z(t) = z[t, \underline{X}(t;\underline{\Sigma})] = \breve{z}(\underline{\Sigma})c[j(t)]^{-1/2}$

The (finite) value of $\breve{z}$ must be obtained by an "indirect method" from the data of the original problem for (1). (See section (2.3).) (The derivation of (21) in [3] is given for the case when M is zero-dimensional, but the argument is valid for any dimension less than n.)

Zero, one, or more rays of the n-parameter family may pass through a given point $(t,\underline{X})$. It is understood that at each point (2) is to be summed over the rays passing through that point. Then (2) is an "asymptotic solution" of (1). The leading term of the asymptotic expansion of the solution of an initial-boundary value problem for (1) then consists of a sum of one or more such asymptotic solutions, so chosen as to satisfy the conditions of the problem. The "ray method" summarized here will be illustrated in sections (2.2), (2.3), (2.5) and (2.6) by applying it to two initial-boundary value problems for (1). In those problems we require the solution of (1) which satisfies the initial conditions

(22) $\qquad u(0,\underline{X}) = u_o(x_1;\ \lambda); \quad u_t(0,\underline{X}) = u_1(x_1;\lambda).$

and the boundary condition

(23) $\ u = 0$ on B.

The functions, u_o and u_1, will be specified. The boundary B will be the parabola

(24) $x_2^2 = -4p(x_1-p).$

In terms of the polar coordinates, $\rho = \sqrt{x_1^2 + x_2^2}$, $\theta = \text{arc tan } x_2/x_1$,

(24) becomes

(25) $\rho(\theta) = \dfrac{2p}{1+\cos\theta}$.

We seek the solution u in the exterior of the parabola, defined by

$\rho \geq 2p(1+\cos\,\theta)^{-1}$. The initial data is assumed to vanish at the boundary.

The solution of each problem consists of a sum of solutions,

(26) $u = u_P + u_R.$

The "primary term" u_P is chosen so as to satisfy the initial conditions of the problem (i.e., the initial conditions for u determine the values, s_o, z_o, etc. for u_P on M_P which is now the initial plane, $t = 0$). The "reflected term" u_R is then constructed in such a way that u satisfies the boundary condition. Hence $u_R = -u_P$ on B. (This determines the initial values, s_o, z_o, etc. for u_R on M_R which is now the boundary.) Of course the initial conditions must not be "spoiled" by u_R. This means that the reflected rays, which emanate from the boundary, must not intersect the initial plane. We shall see that these conditions are just enough to determine u_R uniquely.

(2.2) The Primary Field for Oscillatory Initial Data

Let us consider initial data that is already of the form of (2), that is

$$u(0,\underline{X}) \sim z_o(x_1) \exp[i\lambda s_o(x_1)], \quad u_t(0,\underline{X}) \sim z_1(x_1)\exp[i\lambda s_1(x_1)].$$

This will be called oscillatory initial data [5]. It is shown in [3] that this problem can be solved if we can find the solution to the simpler problem where

$$(27) \quad u(0,\underline{X}) \sim z_o(x_1)\exp[i\lambda s_o(x_1)], \quad u_t(0,\underline{X}) \sim -i\lambda h_o[|s_o'(\sigma_1)|]z_o(x_1)\exp[i\lambda s_o(x_1)],$$

with s_o and z_o given.* Then by comparing (27) and (2) and using (12) and (13) we find that on the initial plane M_P,

$$(28) \quad x_1 = \sigma_1, \; x_2 = \sigma_2, \; t = 0; \; s_o = s_o(\sigma_1), \; z_o = z_o(\sigma_1), \; k_{10} = s_o'(\sigma_1), \; k_{20} = 0.$$

Also, from (20) and (27)

$$(29) \qquad \omega_o = \sqrt{c^2[s_o'(\sigma_1)]^2 + b^2} = h_o[|s_o'(\sigma_1)|].$$

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* We use the symbols s_o and z_o here for given functions and in section (2.1) for the initial values on rays. Since the given functions will immediately be defined as the initial values on primary rays, this double meaning should cause no difficulty.

From (5) and (7) it follows that the primary rays are described by the equations

$$(30)\quad x_1 = \sigma_1 + g_1(\sigma_1)t,\ x_2 = \sigma_2;\ g_1 = c^2 s_o'(\sigma_1)h_o^{-1}[|s_o'(\sigma)|].$$

By using (8) and (9) we find that the phase is given by

$$(31)\quad s = s_o(\sigma_1) + \ell(\sigma_1)t;\quad \ell = -b^2 h_o^{-1}[|s_o'(\sigma_1)|].$$

$z(t)$ is defined by (10). We take $t_o = 0$ and calculate j, defined by (11), from (29). The result is

$$(32)\quad z(t) = z_o(\sigma_1)\ [1+c^2 b^2 s_o''\ h_o^{-3} t]^{-1/2}.$$

Let us assume that $s_o'' > 0$. Then for $t \geq 0$ the jacobian does not vanish and therefore there are no caustics for u_p. We now use (31) and (32) in (2) and express $u_p(t,\underline{X})$ parametrically with parameters (σ_1,σ_2) by (29) and

$$(33)\quad u_p \backsim z_o(\sigma_1)[1+ c^2 b^2 s_o''\ h_o^{-3} t]^{-1/2} \exp\left[i\lambda \left\{ s_o(\sigma_1) - b^2 h_o^{-1} t \right\}\right].$$

As noted in the introduction, to find $u_p(t,\underline{X})$ we add up all contributions of the form (32) from rays passing through $(t,\underline{X})$.

(2.3) The Primary Field for Rapidly Varying Initial Data

We now consider initial data of the form

$$(34)\quad u(0,\underline{X}) = u_o[\lambda(x_1-x_0)],\ u_t(0,X) = \lambda u_1[\lambda(x_1-x_0)];\ x_o > p$$

and assume that $u_o(\xi)$ and $u_1(\xi)$ have compact support.* Then as $\lambda \to \infty$ the support* of the initial data shrinks to the line $L\,(t=0,\ x_1 = x_o)$. In this case we assume that the rays emanate from L** (i.e. $L = M_p$) and that s is constanton L (i.e. independent of $x_2 = \sigma_2$). Then (19) holds. The choice of sign in the dispersion relation (4) is not dictated by the initial data. Since equation (1) is linear, we may take

$$(35)\quad u_p = u_+ + u_-;\quad u_\pm \sim z^\pm \exp[i\lambda s^\pm],$$

where the choice of sign is to be identified with the choice in (4). We apply the method outlined in the introduction for the case of a lower dimensional manifold M and find that the rays, $s^\pm$ and $z^\pm$ are given by

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* The support of a function is the range of values for which the function is not zero. A function is said to have compact support if its support is bounded.

** This assumption is intuitively reasonable. It is fully justified when the solution obtained is compared to the asymptotic expansion of the exact solution and they are found to agree.

(36) $x_1 = x_0 \pm c^2 k_1 h_0^{-1}\, t,\ x_2 = \sigma_2;$

(37) $s^{\pm} = s_0^{\pm} \mp b^2 h_0^{-1} t;$

(38) $z^{\pm} = \zeta^{\pm}\, t^{-1/2}\ .$

Here we have used equations (5), (8), (11) and (21) with $\zeta^{\pm} t^{-1/2} = \tilde{z}^{\pm}(\underline{\Sigma})[j(t)]^{-1/2}$. The constant $s_0^{\pm}$ and the function $\zeta^{\pm}$ are determined by the <u>indirect method</u> discussed in [3]: We first obtain the exact solution of the initial value problem (1), (33) (with no boundary). The asymptotic expansion of this solution is then compared with our result u_P given by (35)—(37) and $s_0^{\pm}$ and $\zeta^{\pm}$ can be identified. The results obtained by this method are

(39) $s_0^{\pm} = 0,\quad \zeta^{\pm} = \left(\frac{2\pi}{\lambda}\right)^{1/2} a_{\pm}(\sigma_1) e^{\mp \frac{i\pi}{4}} \left(h_0''(\sigma_1)\right)^{-1/2}\ ;$

(40) $a_{\pm}(\sigma_1) = \frac{1}{4\pi} \int_{-\infty}^{\infty} [u_0(\xi) \pm \frac{i}{h_0(\sigma_1)}\ u_1(\xi)] \exp[-i\sigma_1 \xi] d\xi,$

(41) $h_0''(\sigma_1) = c^2 b^2 h_0^{-3}\ .$

Then $u_P(t,\underline{X})$ is expressed parametrically with parameters $x_2 = \sigma_2$, $k_1 = \sigma_1$ by (35) and

(42) $u_P \backsim \sum_{\pm} \zeta^{\pm}\, t^{-1/2} \exp[\mp i\lambda b^2 h_0^{-1}\ t].$

(2.4) The Reflected Field at the Boundary

In this section we shall show how the values of s_o, z_o, ω_o, $\underline{K}_o$ and $\underline{G}_o$ for u_R are determined on the boundary M_R. For parameters, we choose $\underline{\Sigma} = (\tau, \theta)$ and using (24), describe M_R parametrically by the equations

$$(43) \quad t = \tau, \quad \rho(\theta) = \frac{2p}{1+\cos\theta}\ ; \quad \underline{X}(\theta) = \rho(\theta)(\cos\theta, \sin\theta).$$

$$0 \leq \tau, \ -\pi \leq \theta \leq \pi.$$

With u_P and u_R of the form in (2) we find, by applying the boundary condition (22), that*

$$(44) \quad z_P[\tau, \underline{X}(\theta)] = -z_R[\tau, \underline{X}(\theta)];$$

$$(45) \quad s_P[\tau, \underline{X}(\theta)] = s_R[\tau, \underline{X}(\theta)].$$

Differentiation of (45) with respect to τ and θ reveals that

$$(46) \quad \omega_P(\tau, \theta) = \omega_R(\tau, \theta),$$

$$(47) \quad k_{1P}(\tau,\theta)\,\dot{x}_1 + k_{2P}(\tau,\theta)\dot{x}_2 = k_{1R}(\tau,\theta)\dot{x}_1 + k_{2R}(\tau,\theta)x_2\ ,$$

where $(\dot{\ })$ denotes differentiation with respect to θ. Then from (46)

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* The subscripts P and R are used throughout this discussion to differentiate between quantities associated with the primary field u_P and the reflected field u_R.

and (4), we conclude that

$$(48) \quad k_P^2 = k_R^2 .$$

We define $\alpha_P(\tau,\theta)$ as the <u>angle of incidence</u> of a primary ray. It is the angle between $\underline{K}_P$ and the normal to B at the point $(\tau,\underline{X}(\theta))$. We define α_R in an analogous manner as the <u>angle of reflection</u>. (47) and (48) show that $\underline{K}_P$ and $\underline{K}_R$ have the same length and projection on the tangent vector to B, $\dot{\underline{X}}(\theta)$. It follows that $\cos \alpha_P = \cos \alpha_R$. If also $\sin \alpha_P = \sin \alpha_R$, the reflected rays would coincide with the primary rays. (See figure (1).) In this case they would intersect the initial plane and u_R would be non-zero for $t = 0$. Then $u = u_P + u_R$ would no longer satisfy the initial conditions. Thus we conclude that $\sin \alpha_P = -\sin \alpha_R$.

Equations (19) and (28) show that (for both types of initial data), $k_{2P} = 0$, i.e. the primary rays are parallel to the axis of the parabola. Then it is easy to show that

$$(49) \quad \underline{K}_R(\tau,\theta) = -k_{1P}(\cos\theta, \sin\theta),$$

that is, the reflected rays would intersect the focus of the parabola if extended backwards. This is a well know property of the parabola. Finally, (7), (46) and (49) yield

$$(50) \quad \underline{G}_R(\tau,\theta) = -c^2 k_{1P}(\tau,\theta)\, \omega_P^{-1}(\tau,\theta)(\cos\theta, \sin\theta).$$

(2.5) The Reflected Field for Oscillatory Initial Data

We consider again the problem introduced in section (2.1) with the initial data of section (2.2). For values of σ_1 for which $s_o'(\sigma_1) < 0$, the primary rays given by (30) will be incident at the boundary. By using (30) and (43) we find that

$$(51)\quad x_1(\tau, \theta) = \frac{2p}{1+\cos\theta} = \sigma_1 + g_1\tau;\ x_2(\tau,\theta) = \frac{2p\sin\theta}{1+\cos\theta} = \sigma_2;$$

$$g_1 = c^2 s_o'(\sigma_1) h_o^{-1}[|s_o'(\sigma_1)|]$$

We recall that our assumption $s_o'' > 0$ led to the conclusion that the jacobian defined by (11) and (30) does not vanish. It follows that (50) always has a unique solution for (σ_1,σ_2) in terms of (τ,θ). Thus each reflected ray is associated with a unique primary ray. By using (5), (50) and (51) we find that

$$(52)\quad X = \rho(\cos\theta, \sin\theta);\quad \rho = \frac{2p}{1+\cos\theta} - \frac{c^2 s_o'(\sigma_1)}{h_o[|s_o'(\sigma_1)|]}\cdot(t-\tau).$$

Using (8), (31) and (45) it follows that

$$(53)\quad s_R = s_p(\tau,\theta) - b^2 h_o^{-1}(t-\tau) = s_o(\sigma_1) - b^2 h_o^{-1}\, t\ .$$

The jacobian j_R, defined by (11) with $\underline{\Sigma} = (\tau,\theta)$, can be calculated from (52). The quotient needed for (10) is given by

$$(54)\quad \frac{j_R(\tau)}{j_R(t)} = \frac{2p}{\rho(1+\cos\theta)} \left[\frac{1+c^2b^2s_o''h_o^{-3}\tau}{1+c^2b^2s_o''h_o^{-3}t}\right] .$$

We use (10), (32) and (44) to determine z_R. This yields

$$(55)\quad z_R = -z_o(\sigma_1)\left[\frac{2p}{\rho(1+\cos\theta)}\right]^{1/2}\left[1+c^2b^2s_o''h_o^{-3}t\right]^{-1/2} .$$

When our results are substituted in (2) u_R is given parametrically with parameters (τ,θ) by (52) and

$$(56)\quad u_R \sim -z_o(\sigma_1)\left[\frac{2p}{\rho(1+\cos\ \theta)}\right]^{1/2}\left[1+c^2b^2s_o''h_o^{-3}t\right]^{-1/2}\exp\left[i\lambda\left\{s_o(\sigma_1)-b^2h_o^{-1}t\right\}\right] .$$

Here σ_1 must be determined from (51) as a function of (τ,θ).

(2.6) The Reflected Field for Rapidly Varying Initial Data.

We consider again, the problem of section (2.1) with the initial data of section (2.3). The only primary rays incident at the boundary are those for which g_1, given by (30), is negative. By using (36) and (43) we find that the analogue of equation (50) for this example is

$$(57)\quad x_1(\tau,\theta) = \frac{2p\cos\theta}{1+\cos\theta} = x_0 \pm \frac{c^2 k_1}{h_0}\tau, \ \pm k_1 < 0; \ x_2 = \frac{2p\sin\theta}{1+\cos\theta} = \sigma_2 .$$

This equation always has a solution forthe parameters (k_1, σ_2) of the primary field in terms of (τ, θ), the parameters of the reflected field.

The calculation of u_R can now be carried out exactly as in section (2.5) and therefore we omit the details. The rays are given by

$$(58)\quad \underline{X} = \rho(\cos\theta, \sin\theta); \quad \rho = \frac{2p}{1+\cos\theta} + \frac{c^2|k_1|}{h_0}(t-\tau).$$

$u_R(t,\underline{X})$ is given parametrically by (58) and

$$(59)\quad u_R \sim \sum_{\pm} \zeta^{\pm}\left[\frac{2p}{\rho(1+\cos\theta)t}\right]^{1/2} \exp\left[\mp i\lambda b^2 h_0^{-1}(k_1)t\right] .$$

Here $\zeta^{\pm}$ are given by (38)-(40), h_0 is given by (6) and k_1 can be calculated. as a function of (τ,θ) from (57).

(3) Asymptotic Expansion of the Exact Solution

(3.1) Derivation of the Solution

In this section we again consider the initial boundary value problems discussed in section (2). Our goal is to obtain the leading term of the asymptotic expansion of the exact solution in order to compare it with the results obtained by the ray method. As in section (2) we assume that $u = u_P + u_R$. The primary field u_P is the solution of an initial-value problem (without a boundary). The reflected field u_R is the solution of an initial-boundary value problem, with zero initial data and boundary data chosen in such a way that $u = 0$ on B.

To find u_P and u_R, we first separate variables by applying Fourier transform with respect to t. This reduces the problem to solving partial differential equations in $\underline{X}$ for functions which will be called v_P and v_R. When these new functions are known, their inverse transforms yield u_P and u_R, expressed as integrals. By expanding these integrals asymptotically, we obtain the asymptotic expansion of $u(t,\underline{X})$.

For both u_P and u_R we define

(1) $$v(\underline{X},\omega) = \int_0^\infty u(t,\underline{X})\exp[i\lambda\omega t]dt; \quad \text{Im } \omega \geq 0$$

and then

(2) $$u(t,\underline{X}) = \sum_{\pm} \frac{\lambda}{2\pi}\int_0^\infty v(\underline{X},\pm\omega)\exp[\mp i\lambda\omega t]d\omega.$$

u_P is a solution of the initial value problem (2.1), (2.21) and is independent of x_2 since (2.21) is independent of x_2. In this case v_P is a function only of x_1. By taking the Fourier transform of (2.1) and

using (2.21), it follows that v_P must satisfy the equation

(3) $$v_P''(x_1,\omega)+\lambda^2 c^{-2}(\omega^2- b^2)v_P(x_1,\omega) = c^{-2}r(x_1,\omega;\lambda);$$

(4) $$r(x_1,\omega;\lambda) = i\lambda\omega u_o(x_1;\lambda) - u_1(x_1;\lambda).$$

v_P also satisfies the "radiation condition,"

(5) $$\lim_{|x_1|\to\infty} v_P(x_1,\omega) = 0; \ \mathrm{Im}\ \omega > 0,$$

which can be derived as in [6] under the assumption that u_o and u_1 have compact support in x_1. Analogous to (3), v_R must satisfy the equation

(6) $$\Delta\, v_R(\underline{X},\omega) + \lambda^2 c^{-2}(\omega^2- b^2)v_R(\underline{X},\omega) = 0; \ \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

The boundary B is given by (2.23) or (2.24) and the boundary condition (22) leads to

(7) $$v_P + v_R = 0 \quad \text{on B.}$$

When v_P is known, (7) defines v_R on B. Analogous to (5) we have the condition

(8) $$\lim_{\rho\to\infty} v_R(\underline{X}, \omega) = 0; \ \mathrm{Im}\ \omega > 0: \ \rho = (x_1^2 + x_2^2)^{\frac{1}{2}}$$

Since (3) is an ordinary differential equation with constant coefficients the solution to (3), (4), (5) can immediately be determined.

The result for $\omega^2 > b^2$ is

(9) $$v_P(x_1,\pm\omega) = \pm \frac{1}{2i\lambda c\sqrt{\omega^2 - b^2}} \left[\int_{-\infty}^{x_1} r(\sigma,\pm\omega;\lambda)\exp[\pm i\lambda k(x_1-\sigma)]d\sigma \right.$$

$$\left. + \int_{x_1}^{\infty} r(\sigma,\pm\omega;\lambda)\exp[\mp i\lambda k(x_1-\sigma)]d\sigma \right];$$

$$ck = \sqrt{\omega^2 - b^2} .$$

The solution of (9) for $\omega^2 < b^2$ could be obtained by analytic continuation of (9), but is of no interest since it is exponentially small in λ and leads to an exponentially small contribution to u_P. At the boundary, the first integral in (9) is zero, since the entire domain of integration is outside the support of $r(\sigma,\omega;\lambda)$. Therefore, for $\omega > b$,

(10) $$v_P(x_1,\pm\omega) = A(\pm\omega)\ \exp[\mp i\lambda k x_1] \quad \text{on } B,$$

where

(11) $$A(\pm\omega) = \pm \frac{1}{2i\lambda c\sqrt{\omega^2 - b^2}} \int_{-\infty}^{\infty} r(\sigma,\pm\omega;\lambda)\exp[\pm i\lambda k\sigma]d\sigma.$$

In order to solve (6) - (8) we use the method of Lamb [6]. We assume that $v_R = w\ \exp[\mp i\lambda k x_1]$. The function w can then be determined in terms of parabolic coordinates (ξ,η), which are defined by the equation

(12) $$(\xi + i\eta)^2 = \lambda k(x_1 + i\ x_2).$$

In terms of the polar coordinates, (ρ,θ), it can be shown that

(13) $$\xi = (\lambda k\rho)^{\frac{1}{2}} \cos\frac{\theta}{2}\ ,\quad \eta = (\lambda k\rho)^{\frac{1}{2}} \sin\frac{\theta}{2}\ .$$

From (2.24), the boundary is now defined by the equation

$$(14) \qquad \xi = \xi_B = (\lambda k p)^{\frac{1}{2}}.$$

The calculation of v_R is carried out in Appendix I.

The result is

$$(15) \qquad v_R(\underset{\sim}{X}, \pm\omega) = -A(\pm\omega)\exp[\mp i\lambda k x_1]\left\{E^{\pm}[(\lambda kp)^{\frac{1}{2}}]\right\}^{-1}\left\{E^{\pm}[(\lambda k\rho)^{\frac{1}{2}}\cos\frac{\theta}{2}]\right\};$$

$$(16) \qquad E^{\pm}(\xi) = \int_{\xi}^{\infty} \exp[\pm 2i\zeta^2]d\zeta.$$

This expression can be simplified by expanding $E^{\pm}$ asymptotically for large λ. By integrating by parts in (16), we obtain

$$(17) \qquad E^{\pm}(\xi) \sim \frac{\mp \exp[\pm 2i\xi^2]}{4i\xi}$$

When this result and the half angle formula for $\cos\frac{\theta}{2}$ are substituted in (15), v_R can be rewritten as

$$(18) \qquad v_R(\underline{X}, \pm\omega) \sim -A(\pm\omega)\left(\frac{2\rho}{\rho(1+\cos\theta)}\right)^{\frac{1}{2}} \exp[\pm i\lambda k(\rho - 2p)].$$

(3.2) <u>A Sample Solution</u>

With v_P and v_R given by (9) and (18), u_P and u_R can be determined by substituting in (2) and then expanding asymptotically by the method of stationary phase. This has been carried out for the two problems considered in section (2) and in both cases the results agree exactly with those obtained by the ray method. As an example we consider u_R for the problem with oscillatory initial data (2.26). Using that equation and (4) we find that

$$(19) \qquad r(x_1,\ \pm\omega;\lambda) = i\lambda z_o(x_1)\left[\pm\omega + h_o\left(|s_o'(x_1)|\right)\right]\exp[i\lambda s_o(x_1)].$$

When this value is substituted into (11),

$$(20) \quad A(\pm\omega) = \pm\frac{1}{2c\sqrt{\omega^2-b^2}}\int_{-\infty}^{\infty} z_o(\sigma)\left[\pm\omega + h_o\left(|s_o'(\sigma)|\right)\right]\exp[i\lambda\{s_o(\sigma)\pm k\sigma\}]d\sigma,$$

$$ck = \sqrt{\omega^2-b^2}$$

and then $v_R(\underline{X},\pm\omega)$ is given by (18) with this value of $A(\pm\omega)$. $u_R(t,\underline{X})$ is now given asymptotically by (2) with v replaced by v_R:

$$(21) \quad u_R(t,\underline{X}) \sim \sum_{\pm} \mp \frac{\lambda}{4\pi c^2}\left(\frac{2p}{\rho(1+\cos\theta)}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}d\sigma\int_0^{\infty}d\omega\left[\frac{z_o(\sigma)\left[\pm\omega + h_o\left(|s_o'(\sigma)|\right)\right]}{k}\right.$$

$$\left.\cdot\exp\left[\pm i\lambda\{k(\rho-2p+\sigma)-\omega t\} + i\lambda\, s_o(\sigma)\right]\right].$$

This integral can be evaluated by the method of stationary phase in two dimensions [3]. We define

$$(22) \quad \varphi_{\pm}(\omega,\sigma) = \pm\left\{\frac{\sqrt{\omega^2-b^2}}{c}(\rho+\sigma-2p) - \omega t\right\} + s_o(\sigma).$$

It is necessary to determine the first and second derivatives of $\varphi_{\pm}$ with respect to ω and σ; these functions are:

$$(23) \quad \frac{\partial\varphi_{\pm}}{\partial\omega} = \pm\left\{\frac{\omega}{c\sqrt{\omega^2-b^2}}(\rho+\sigma-2p) - t\right\}; \quad \frac{\partial\varphi_{\pm}}{\partial\sigma} = \pm\frac{\sqrt{\omega^2-b^2}}{c} + s_o'(\sigma).$$

$$(24) \quad \frac{\partial^2\varphi_{\pm}}{\partial\omega^2} = \mp\left\{\frac{b^2}{c(\omega^2-b^2)^{3/2}}(\rho+\sigma-2p)\right\}; \quad \frac{\partial^2\varphi_{\pm}}{\partial\sigma^2} = s_o''(\sigma);$$

(25) $$\frac{\partial^2 \varphi_\pm}{\partial \omega \, d\sigma} = \pm \frac{\omega}{c\sqrt{\omega^2 - b^2}} \quad .$$

The stationary points of $\varphi_\pm$ are determined by setting both derivatives in (23) equal to zero. This yields the equations

(26) $$(\rho+\sigma-2p) = \frac{c\sqrt{\omega^2 - b^2}}{\omega}\, t; \quad c\, s_o'(\sigma) = \mp\sqrt{\omega^2 - b^2} \quad .$$

By solving for ω we find that

(27) $$\omega = c^2 s_o'^2(\sigma) + b^2 = h_o[\,|\, s_o'(\sigma)\,|\,].$$

Hence, for the lower sign in (21) the integrand is zero. For the upper sign, from (26), it is necessary that $s_o'(\sigma) < 0$. For the domain of the problem $\rho \geq p$. In the support of z_o, $\sigma > p$. Therefore in the domain where the integrand is non-zero, $\rho + \sigma - 2p > 0$, and the first equation of (26) can always be satisfied for $t > 0$. If we define the boundary in space time by (2.25) and $t = \tau$ then at the boundary (26) becomes

(28) $$\left(\frac{2p}{1+\cos\theta} + \sigma - 2p\right) = \frac{c\sqrt{\omega^2 - b^2}}{\omega}\, \tau.$$

Multiplying (27) by $\cos\theta$ yields the first equation of (2.51) which defines σ (or σ_1 of section (2)) as a function of (τ, θ). The equation for $\sigma_2 = x_2 = \rho(\theta) \sin\theta$ in (2.51) is an immediate consequence of (2.25). We see from (26), (27) and (28) that the stationary phase conditions are equivalent to the equation of the rays (2.52), the dispersion relation (2.29) and the relation between the parameters (σ_1, σ_2) and (τ, θ).

We define

$$(29) \qquad (\partial^2\varphi) = \begin{pmatrix} \varphi_{\omega\omega} & \varphi_{\omega\sigma} \\ \varphi_{\sigma\omega} & \varphi_{\sigma\sigma} \end{pmatrix}$$

where all derivatives are evaluated at the stationary point defined by (28). Then

$$(30) \qquad \det(\partial^2\varphi_+) = -\frac{h_o^2}{c^4 s_o'^2}\left[1 + \frac{c^2 b^2 s_o''}{h_o^3}\, t\right].$$

For $s_o'' > 0$, the eigenvalues of $\partial^2\varphi$ must be of opposite sign for all $t > 0$ and therefore sig $(\partial^2\varphi) = 0$. Evaluating φ at the stationary point, we find that

$$(31) \qquad \varphi(\omega,\sigma) = -\frac{b^2}{h_o}\, t + s_o(\sigma).$$

The solution is now obtained by using our results in the formula in [3] for evaluation of integrals by the method of stationary phase in two dimensions. $u_R(t,X)$ is given parametrically by (26), (27), and

$$(32) \qquad u_R \sim -\left(\frac{2p}{\rho(1+\cos\theta)}\right)^{\frac{1}{2}} \cdot \frac{z_o(\sigma)}{[1+c^2b^2s_o''h_o^{-3}t]^{\frac{1}{2}}} \cdot \exp[i\lambda\{s_o(\sigma) - b^2h_o^{-1}t\}].$$

This is the same result as was obtained in section (2.5) by the ray method.

(4) Numerical Comparison of the Exact and Asymptotic Solutions

We consider the problem for u defined in section (2.1) in the special case where

(1) $$u(0, \underline{X}) = 0, \quad u_t(0, \underline{X}) = \delta(x_1 - 2p).$$

In this section we shall compare numerically the exact solution and asymptotic solutions u_R for some region of space-time. It is most convenient and probably most interesting to make this comparison on the axis of the parabola for $x_1 \gg 1$ and $t \gg 1$; i.e., we take $x_2 = 0$, $x_1 = \rho = n{\cdot}ct$ $(0 < n < 1)$, $\theta = 0$ and compare the exact and asymptotic solutions as $t \to \infty$. Let us define $\tilde{u}_R$ to be the asymptotic solution obtained by the ray method. ($u_R \sim \tilde{u}_R$ for $\lambda \to \infty$.) We also define $\hat{u}_R$ to be the expansion of the exact solution for large values of t. ($u_R \sim \hat{u}_R$ as $t \to \infty$, $x_2 = 0$, $x_1 = nct$.) In addition we introduce $\hat{\tilde{u}}_R$ as the expansion of $\tilde{u}_R$ as $t \to \infty$. $\hat{\tilde{u}}_R$ and $\hat{u}_R$ will be compared for a typical value of n as λ varies.

We note that $\delta(x_1 - 2p) = \lambda\delta(\lambda[x_1 - 2p])$ so that (1) is a special case of rapidly varying initial data. $\tilde{u}_R$ is given parametrically by (2.58) and (2.59). $\zeta^{\pm}$ are obtained by using (1), (2.34), (2.39) and (2.40). It then follows that

(2) $$\tilde{u}_R(t,\rho,0) \sim \sum_{\pm} - \left(\frac{ph_o}{8\pi\lambda c^2 b^2 \rho t}\right)^{1/2} \exp\left[\mp i\lambda b^2 h_o^{-1} t \pm \frac{i\pi}{4}\right];$$

(3) $$\rho = p + \frac{c^2|k_1|}{h_o}(t - \tau).$$

The dependence on ρ in (2) can be eliminated by making use of (3). Before doing so, we note from (2.57) with $x_o = 2p$ that

(4) $$p = \frac{c^2|k_1|}{h_o}\tau ,$$

so that (3) becomes

(5) $$\rho = nct; \quad n = \frac{c|k_1|}{h_o} < 1.$$

Then (2) may be rewritten as

(6) $$\tilde{u}_R(t, vt, 0) \sim \sum_{\pm} - \left(\frac{ph_o}{8\pi\lambda b^2 c^3 nt^2}\right)^{\frac{1}{2}} \exp\left[\mp i\lambda b^2 h_o^{-1} t \pm \frac{i\pi}{4}\right] .$$

To find $\hat{u}_R$ we must expand the coefficient of the exponent in (6) in powers of $1/t$. But we see at once that this coefficient is already of the form const. $\frac{1}{t}$. Therefore $\hat{\tilde{u}}_R = \tilde{u}_R$.

The exact expression for u_R can be obtained by using (3.15) with $\Theta = 0$ and $x_1 = \rho$. $A(\pm\,\omega)$ is defined by (3.11), where

(7) $$r(\sigma, \pm\,\omega; \lambda) = -\delta(x_1 - 2p)$$

as determined by (1) and (3.4). We seek the value of v_R for large ρ so that $E^{\pm}([\lambda k\rho]^{\frac{1}{2}})$, appearing in (3.15) can be expanded asymptotically for large argument. This expansion can be obtained from (3.17) and the resulting solution for v_R is

(8) $$v_R(\rho, 0, \pm\,\omega) \sim \left(8\lambda c^2 k\sqrt{\lambda k\rho}\;\; E^{\pm}\left(\sqrt{\lambda kp}\right)\right)^{-1} \exp\left[\pm\, i\lambda(k(\rho + 2p))\right];$$

$$ck = \sqrt{\omega^2 - b^2} .$$

Using (3.2) with u replaced by u_R and $\rho = nct$, we find that

$$(9)\quad u_R(t,vt,0) \sim \sum_{\pm} \frac{1}{16\pi c^2} \int_0^{\infty} \frac{\exp\left[\pm i\lambda(knc-\omega)t \pm 2i\lambda kp\right]}{k\sqrt{\lambda knct}\, E^{\pm}(\sqrt{\lambda kp})}\, d\omega,\quad t \to \infty.$$

This integral can be expanded asymptotically by the method of stationary phase for large values of t. The result, $\hat{u}_R$, is given by

$$(10)\quad \hat{u}_R \sim \sum_{\pm} - R_{\pm} \left(\frac{ph_o}{8\pi\lambda b^2 c^3 n t^2}\right)^{\frac{1}{2}} \exp\left[\mp i\lambda b^2 h_o^{-1} t \pm \frac{i\pi}{4}\right];$$

$$n = \frac{ck}{h_o} \leq 1,$$

where

$$(11)\quad R_{\pm} = \mp \frac{\exp\left[\pm 2i\lambda kp\right]}{4i\sqrt{\lambda kp}\, E^{\pm}(\sqrt{\lambda kp})}.$$

Comparing (6) and (11) we see that $\tilde{u}_R$ and $\hat{u}_R$ differ only in the factors $R_{\pm}$. To compare the solutions numerically then, we need only examine $R_{\pm}$ and see how they compare to one. It should be noted that $R_{\pm}$ are complex conjugates and are just ratios of the asymptotic expansions of $E^{\pm}(\sqrt{\lambda kp})$ to the functions themselves. In terms of n, we find that

$$(12)\quad \lambda kp = \frac{\lambda bp}{c} \frac{n}{\sqrt{1-n^2}},$$

so that $R_{\pm}$ are functions of the two (dimensionless) parameters n and $\frac{\lambda bp}{c}$.

As an example, in figure (3) we show a graph of the magnitudes of Re R_+ and for ImR_+ for $n = \frac{1}{2}$ and $\frac{\lambda b p}{c} = \nu$ varying. The table below lists some sample values used to construct the graph.

ν	Re R_+	Im R_+
π	1.0269	.12338
2π	1.0083	.06660
4π	1.0014	.03475
8π	1.0006	.01686
10π	1.0004	.01414

Appendix I

In this section we will show how the function $v_R(\underline{X}, \pm\omega)$, defined in section (3.1), is obtained by Lamb's method. This function must satisfy the equation

$$(1) \qquad \Delta v_R + \lambda^2 c^{-2}(\omega^2 - b^2) v_R = 0$$

We assume that

$$(2) \qquad v_R(\underline{X}, \pm\omega) = w(\underline{X}, \pm\omega) \exp\left[\mp i\lambda k x_1\right]; \qquad ck = (\omega^2 - b^2)^{1/2}$$

Then at the boundary from (3.10)

$$(3) \qquad w(\underline{X}, \pm\omega) = -A(\pm\omega).$$

In order that v_R satisfy (1), w must satisfy the equation,

$$(4) \qquad \Delta w(\underline{X}, \pm\omega) \mp 2i\lambda k w_{x_1} = 0;$$

Following Lamb's example, we introduce the parabolic coordinates, (ξ, η), defined by the equation

$$(5) \qquad (\xi + i\eta)^2 = \lambda k (x_1 + i x_2).$$

In terms of the polar coordinates (ρ, θ), (5) can be reduced to

$$(6) \qquad \xi = (\lambda k \rho)^{1/2} \cos\frac{\theta}{2}, \qquad \eta = (\lambda k \rho)^{1/2} \sin\frac{\theta}{2}.$$

From (2.24), the boundary is now defined by the equation

(7) $$\xi = \xi_B = (\lambda \ k \ p)^{1/2}.$$

For the function

(8) $$W(\xi, \eta, \pm\omega) = w(X(\xi, \eta), \pm\omega),$$

the following equation is obtained from (4):

(9) $$\frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial \eta^2} \mp 4i \left(\xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right) W = 0$$

Since the boundary is given by ξ = constant and the boundary condition, (3), is w = constant (and therefore W = constant), we seek a solution to (9) independent of η. In this case, W must satisfy the equation

(10) $$\frac{d^2 W}{d\xi^2} \mp 4i\xi \frac{dW}{d\xi} = 0.$$

A solution of (10) is

(11) $$W(\xi, \pm\omega) = R_{\pm} E^{\pm}(\xi)$$

where

(12) $$E^{\pm}(\xi) = \int_{\xi}^{\infty} \exp\left[\pm 2i\zeta^2\right] d\zeta.$$

We note that $R_{\pm}$ and ξ are functions of ω. It can be shown that when this function is substituted into (14), v_R satisfies the radiation condition.

Therefore, if $R_{\pm}$ are chosen so as to make W satisfy the boundary condition, (3), v_R will be determined. From (3), (7) and (11), it follows that

(13) $$R_{\pm} = -A(\pm\omega)\left\{ E^{\pm}\left([\lambda\ k\ p]^{1/2} \right) \right\}^{-1}$$

Using (11), (12) and (13) in (2), we obtain

(14) $$v_R(\underline{X}, \pm\omega) = -A(\pm\omega)\exp\left[\mp i\lambda k x_1\right] \left\{ E^{\pm}\left([\lambda k p]^{1/2} \right) \right\}^{-1} \left\{ E\left([\lambda k \rho]^{1/2} \cos \frac{\theta}{2} \right) \right\} .$$

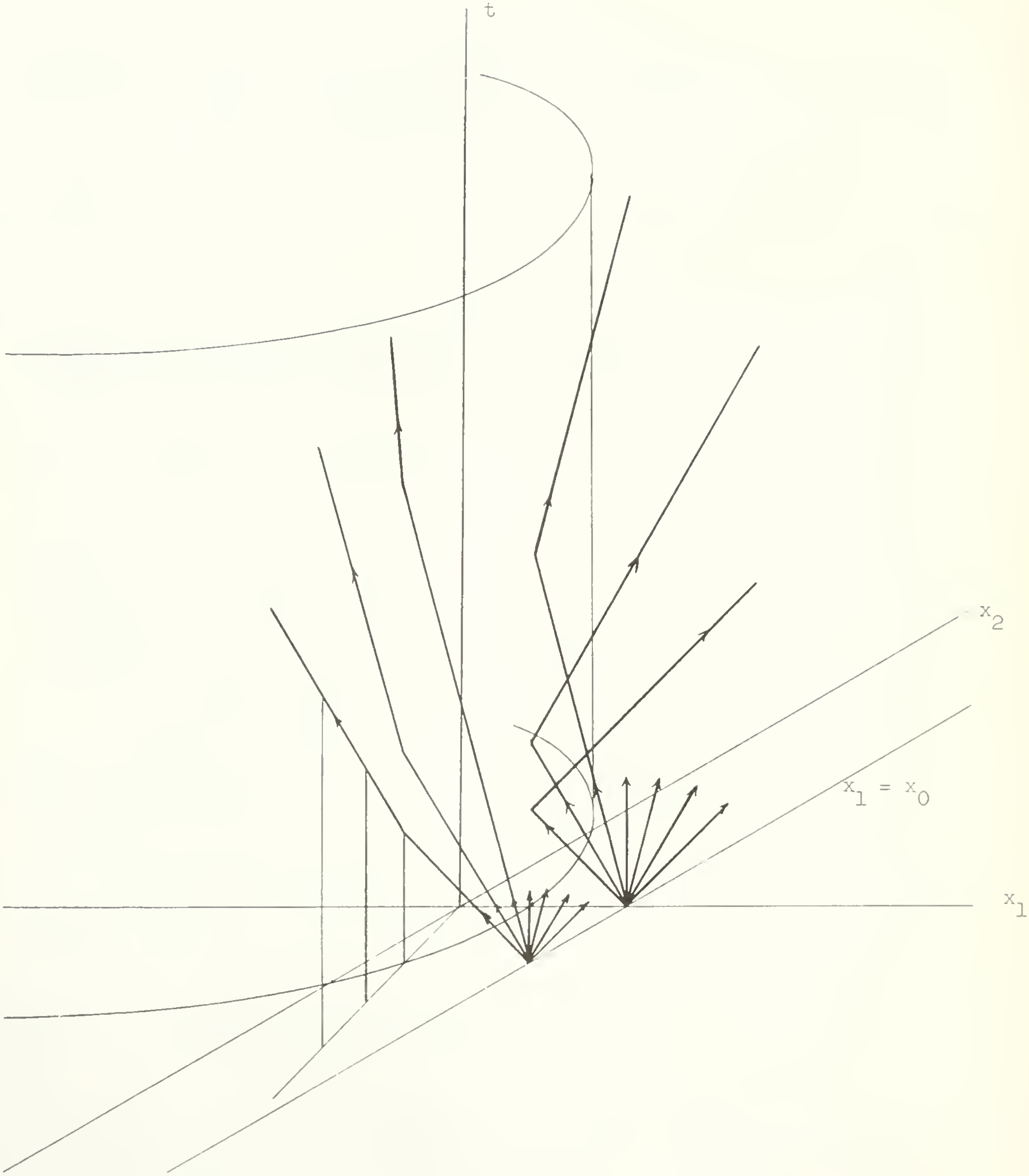

Fig. 1

Ray picture in space-time for the case
of rapidly varying initial data.

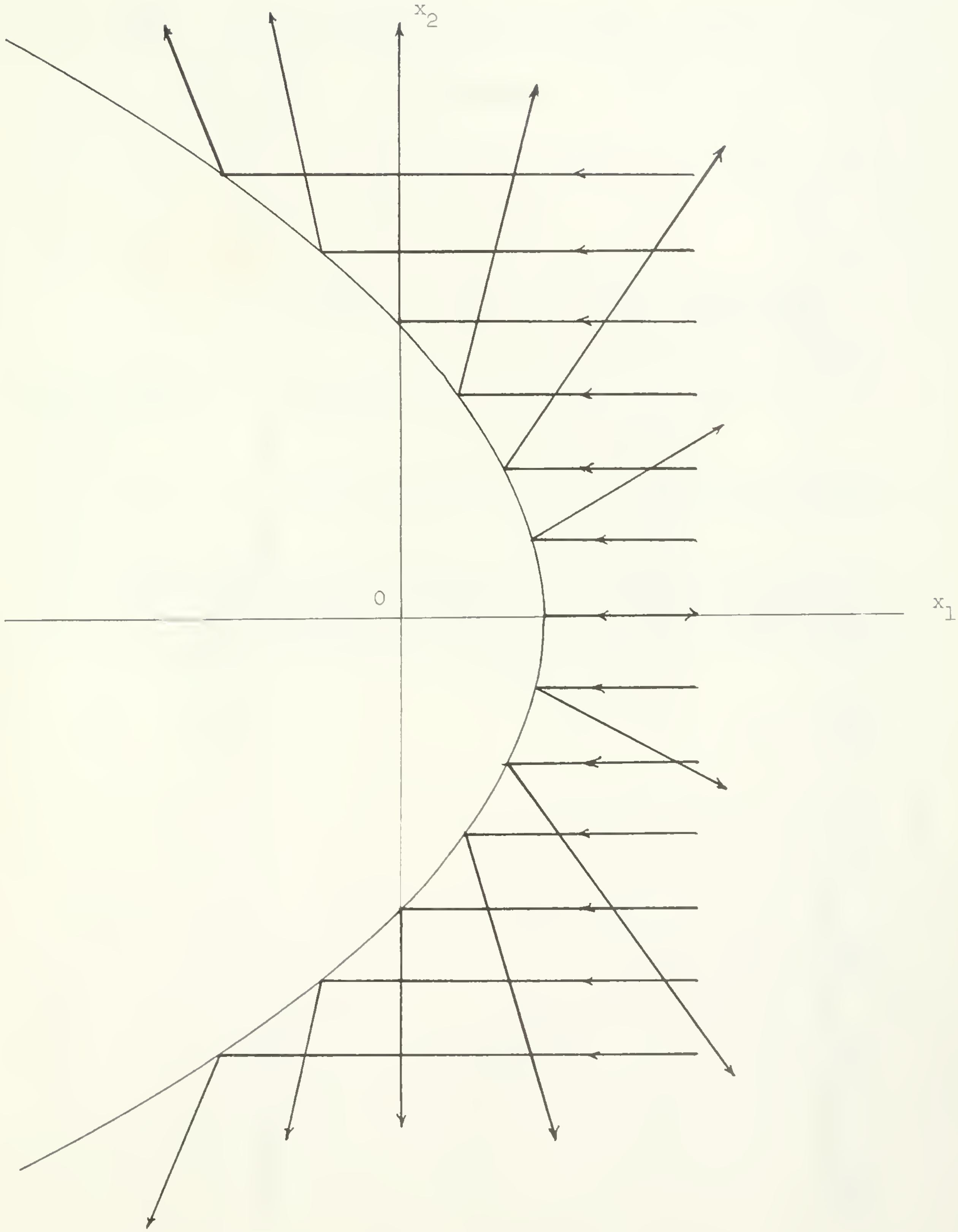

Fig. 2

Projection of rays in space.

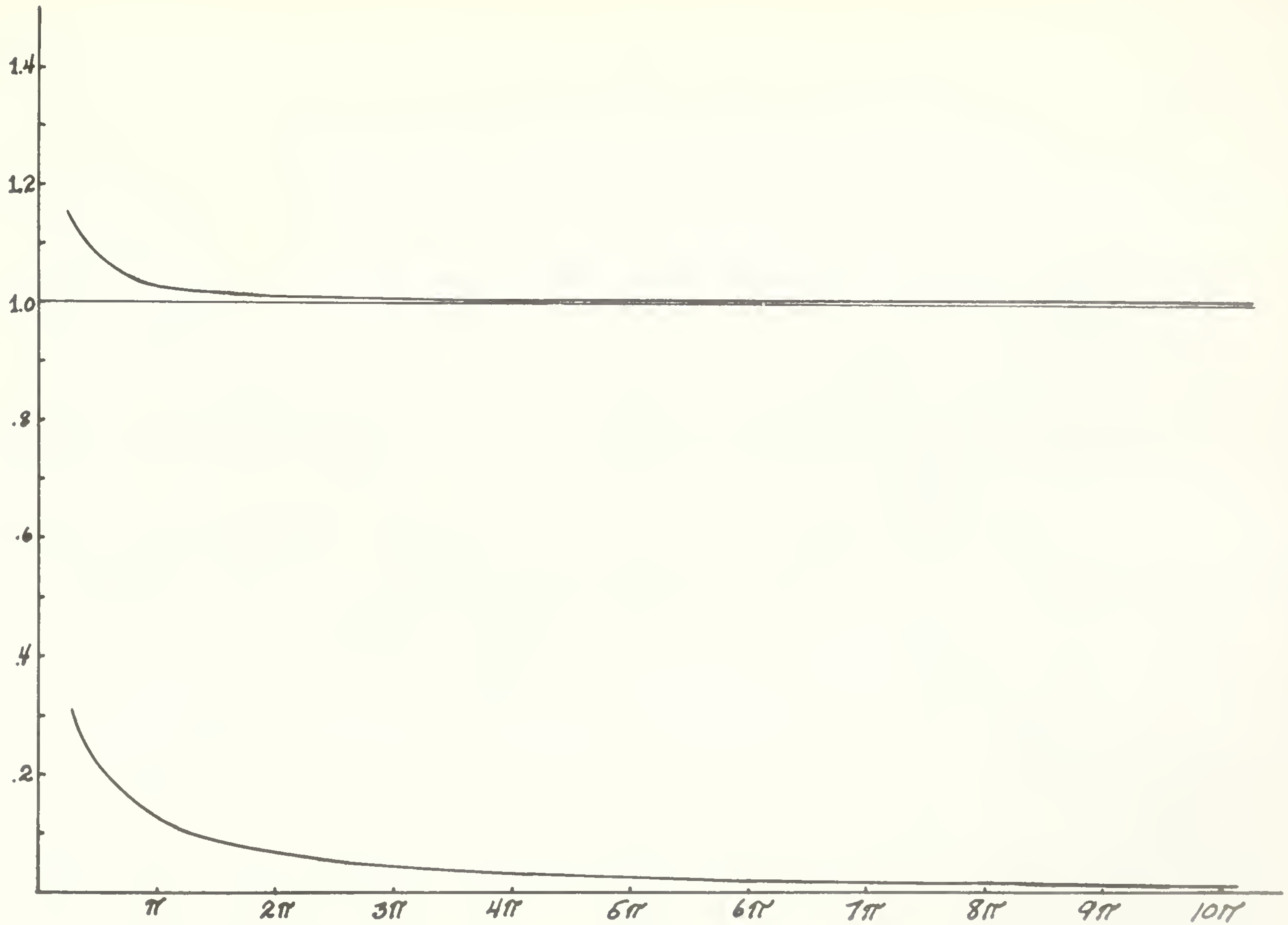

Fig. 3

Graph of $\mathrm{Re}R_+$ (upper curve) and $\mathrm{Im}R_+$ (lower curve) as a function of $\nu = \frac{\lambda bp}{c}$ for $n = \frac{1}{2}$.

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13. ABSTRACT

Initial-boundary value problems for an energy conserving dispersive hyperbolic equation, the Klein-Gordon equation, are considered. This equation exhibits the main feature of dispersion: The speed of propagation depends on frequency. Problems in two space dimensions with a parabolic boundary are discussed.

The primary purpose of this paper is to compare the asymptotic expansion of solutions obtained by a technique we call the ray method with the asymptotic expansion of the exact solution. In the cases considered, the solutions agree.

In addition a numerical comparison is made of the exact and asymptotic solutions for a specified region of space time.

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Attachment 1

14. KEY WORDS	LINK A		LINK B		LINK C	
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